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Harmonic oscillator with strongly pulsating mass

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Abstract. An exact solution is presented for the problem of a harmonic oscillator of frequency ω_0 and mass varying with time according to $M = M_0 \cos^2 \nu t$. The solution is closely related to that of an oscillator of constant mass M_0 and frequency $(\omega_0^2 + \nu^2)^{1/2}$. Pseudostationary and quasi-coherent states are discussed. Applications in quantum optics are foreseen.

1. Introduction

The problem of the damped harmonic oscillator has received much attention over the past decade (Louisell 1973, Hasse 1975, Dodonov and Man'ko 1979). This problem can be regarded as one in which the mass of the oscillator changes exponentially and some authors have considered the idea of a more general change of mass with time (Dodonov and Man'ko 1979, Remand and Hernandez 1980). The present authors have suggested a new approach (Colegrave and Abdalla 1981a, b) in which time-dependent canonical transformation theory is invoked to transform the variable mass oscillator to the standard constant mass form. The case of exponentially changing mass (Colegrave and Abdalla 1981b, hereafter referred to as I) is the simplest case, the time dependence of the mass $M = M_0 \exp(-2\gamma t)$ being transformed away immediately since the fluctuation function

$$\gamma_0(t) = (1/2M)\, \mathrm{d}M/\mathrm{d}t = -\gamma \tag{1.1}$$

is constant. A time-free Hamiltonian remains which is readily reduced to diagonal form, leaving a constant mass harmonic oscillator of reduced frequency ω, where

$$\omega^2 = \omega_0^2 - \gamma^2 \tag{1.2}$$

as shown in I.

We can now report a second and perhaps more interesting solvable case in which the mass is a periodic function of the time as discussed in § 2. Here the Hamiltonian has been reduced by a further time-dependent canonical transformation to that for a harmonic oscillator of constant mass and frequency Ω given by

$$\Omega^2 = \omega_0^2 + \nu^2 \tag{1.3}$$

where ν is the frequency of $M^{1/2}(t)$. Again the time dependence of the solution resides solely in the canonical transformations.

The order of presentation is similar to that adopted in I. We start in § 3 with the discovery of a solution of the Schrödinger equation. After a consideration of the

Heisenberg equations in § 4 and a discussion of the energy in § 5, we are able to make the correct definition of the Dirac operator $A_1(t)$ in § 6, leading to the time-dependent diagonalising transformation in § 7. In the final sections we discuss the Green function, quasi-coherent states and the possibility of solving this type of problem entirely in the Schrödinger picture.

2. The mass law $M = M_0 \cos^2 \gamma t$

As discussed elsewhere (Colegrave and Abdalla 1981a), the electric and magnetic field intensities in a Fabry–Pérot cavity are proportional to $M^{1/2}(t)$, where $M(t)$ is the variable mass parameter in the harmonic oscillator Hamiltonian. Hence the mass law

$$M^{1/2}(t) = M_0^{1/2} \cos \nu t \tag{2.1}$$

describes a harmonically varying field intensity, a situation which could ideally arise (Kumar and Mehta 1981). Although the mass is never negative it would be even better to avoid its periodic vanishing, but a modification of equation (2.1) to this end leads to awkward analysis and any hope of an exact solution is lost. However, we encounter no mathematical difficulty since $M \to 0$ periodically.

3. Solution of the Schrödinger equation

As described in I, we transform the variable mass Hamiltonian

$$H(t) = \tfrac{1}{2}p^2/M(t) + \tfrac{1}{2}M(t)\omega_0^2 q^2, \qquad [q, p] = \mathrm{i}\hbar, \tag{3.1}$$

via the canonical transformation

$$q_0 = [M(t)/M_0]^{1/2} q, \qquad P_0 = [M_0/M(t)]^{1/2} p, \tag{3.2}$$

to a form in which the time dependence is concentrated in the fluctuation function $\gamma_0(t)$ defined in equation (1.1). The new Hamiltonian is

$$H_0(t) = \tfrac{1}{2}p_0^2/M_0 + \tfrac{1}{2}M_0\omega_0^2 q_0^2 + \tfrac{1}{2}\gamma_0(t)(q_0 p_0 + p_0 q_0), \qquad [q_0, P_0] = \mathrm{i}\hbar. \tag{3.3}$$

For the mass law (2.1) the transformation (3.2) is

$$q_0 = q \cos \nu t, \qquad p_0 = p \sec \nu t, \tag{3.4}$$

with

$$\gamma_0(t) = -\nu \tan \nu t. \tag{3.5}$$

The Schrödinger equation corresponding to the Hamiltonian (3.3) is

$$\left(-\frac{\hbar^2}{2M_0}\frac{\partial^2}{\partial q_0^2} + \tfrac{1}{2}M_0\omega_0^2 q_0^2 + \mathrm{i}\hbar\nu \tan \nu t\left(q_0 \frac{\partial}{\partial q_0} + \frac{1}{2}\right)\right)\psi = \mathrm{i}\hbar\frac{\partial \psi}{\partial t}. \tag{3.6}$$

Pseudostationary or quasi-periodic solutions (Gesztesy and Mitter 1981) exist of the form (we revert to the physical coordinate q by the transformation (3.4))

$$\psi_l(q, t) \exp[-\mathrm{i}\Omega(l + \tfrac{1}{2}t], \qquad l = 0, 1, 2, \ldots, \tag{3.7}$$

where

$$\psi_l(q,t)=N_l(t)H_l([M(t)\Omega/\hbar]^{1/2}q)\exp[-M(t)(\Omega-\mathrm{i}\nu\tan\nu t)q^2/(2\hbar)], \tag{3.8a}$$

$$N_l(t)=[M(t)\Omega/(\pi\hbar)]^{1/4}(2^l l!)^{-1/2} \tag{3.8b}$$

and Ω is the augmented frequency given by equation (1.3). It is interesting to compare $\psi_l(q,t)$ with the following damped solution obtained by Hasse (1975) and Tartaglia (1977) and which the present authors obtained in I:

$$\psi_l(q,t)=N_l(t)H_l([M(t)\omega/\hbar]^{1/2}q)\exp[-M(t)(\omega-\mathrm{i}\gamma)q^2/(2\hbar)], \tag{3.9}$$

where $N_l(t)$ is given by equation (3.8*b*) with $M(t)=M_0\exp(-2\gamma t)$. Like (3.9), our present solution (3.7) remains normalised in the original physical coordinate q at all times, except at the instants when $M=0$. At these times $\psi=0$. We notice that (3.7) may be obtained from (3.8) by replacing the reduced frequency ω by the augmented frequency Ω, the constant damping factor γ by the function $-\gamma_0=\nu\tan\nu t$ and the damped mass by the pulsating mass. We hasten to add that this extension is not true, as far as we can see, for any other mass law. For instance, consider as a generalisation of the law (2.1)

$$M(t)=M_0\cos^n\nu t, \tag{3.10}$$

so that

$$\gamma_0(t)=-\nu n\tan\nu t; \tag{3.11}$$

then it may easily be seen that an extension of the wavefunction (3.9) of the type (3.7) is true only for the case $n=2$.

Let us write, as in I,

$$T=\tfrac{1}{2}p^2/M(t)=\tfrac{1}{2}p_0^2/M_0, \qquad V=\tfrac{1}{2}M(t)\omega_0^2q^2=\tfrac{1}{2}M_0\omega_0^2q_0^2. \tag{3.12}$$

The matrix elements of T, V and H_0 may be calculated with respect to the states (3.8) and we obtain

$$\langle l|V|l'\rangle=\tfrac{1}{4}\hbar(\omega_0^2/\Omega)\{(2l+1)\delta_{ll'}+[(l+1)(l+2)]^{1/2}\delta_{l,l'-2}+[(l'+1)(l'+2)]^{1/2}\delta_{l-2,l'}\}, \tag{3.13a}$$

$$\begin{aligned}\langle l|T|l'\rangle=&\tfrac{1}{2}\hbar\Omega(l+\tfrac{1}{2})[1+(\nu/\Omega)^2\tan^2\nu t]\delta_{ll'}\\&-\tfrac{1}{4}\hbar\Omega[(l+1)(l+2)]^{1/2}[1-2\mathrm{i}(\nu/\Omega)\tan\nu t-(\nu/\Omega)^2\tan^2\nu t]\delta_{l,l'-2}\\&+\tfrac{1}{4}\hbar\Omega[(l'+1)(l'+2)]^{1/2}[1+2\mathrm{i}(\nu/\Omega)\tan\nu t-(\nu/\Omega)^2\tan^2\nu t]\delta_{l-2,l'},\end{aligned} \tag{3.13b}$$

$$\begin{aligned}\langle l|H_0|l'\rangle=&\hbar\Omega(l+\tfrac{1}{2})[1-\tfrac{1}{2}(\nu/\Omega)^2\sec^2\nu t]\delta_{ll'}\\&-\tfrac{1}{4}\hbar(\nu^2/\Omega)\sec^2\nu t\{[(l+1)(l+2)]^{1/2}\delta_{l,l'-2}+[(l'+1)(l'+2)]^{1/2}\delta_{l-2,l'}\}.\end{aligned} \tag{3.13c}$$

We notice that $\langle l|V|l'\rangle$ is time independent and equal to ω_0/Ω times the value for an oscillator with constant mass:

$$\langle l|V|l'\rangle=(\omega_0/\Omega)\langle l|V|l'\rangle_{M=M_0}. \tag{3.14}$$

Also, on letting $\nu\to0$ so that $\Omega\to\omega_0$, equation (3.13*b*) gives the correct value for

$$\langle l|T|l'\rangle_{M=M_0}=\tfrac{1}{4}\hbar\omega_0\{(2l+1)\delta_{ll'}-[(l+1)(l+2)]^{1/2}\delta_{l,l'-2}-[(l'+1)(l'+2)]^{1/2}\delta_{l-2,l'}\}. \tag{3.15}$$

The expectation values of V, T, $qp+pq$ and H_0 in the state ψ_l are

$$\langle l|V|l\rangle = \tfrac{1}{2}\hbar\Omega(\omega_0/\Omega)^2(l+\tfrac{1}{2}), \tag{3.16a}$$

$$\langle l|T|l\rangle = \tfrac{1}{2}\hbar\Omega[1+(\nu/\Omega)^2\tan^2\nu t](l+\tfrac{1}{2}), \tag{3.16b}$$

$$\langle l|qp+pq|l\rangle = 2\hbar(\nu/\Omega)\tan\omega t(l+\tfrac{1}{2}), \tag{3.16c}$$

$$\langle l|H_0|l\rangle = \hbar\Omega[1-\tfrac{1}{2}(\nu/\Omega)^2\sec^2\nu t](l+\tfrac{1}{2}). \tag{3.16d}$$

At times $t=n\pi/\nu$, $n=0, 1, 2, \ldots$ when $M=M_0$, the expectation values of $T+V$ and H_0 are equal:

$$\langle l|T+V|l\rangle_{\min} = \langle l|H_0|l\rangle_{\max} = \hbar\Omega[1-\tfrac{1}{2}(\nu/\Omega)^2](l+\tfrac{1}{2}). \tag{3.17}$$

It is interesting to compare the results (3.16a, b) with the expectation values of T and V for the case of decaying mass considered in I, where both expectation values are equal and time independent,

$$\langle l|V|l\rangle = \langle l|T|l\rangle = \tfrac{1}{2}\hbar\omega(\omega_0/\omega)^2(l+\tfrac{1}{2}),$$

ω being the reduced frequency given by equation (1.2). These values, together with the other results obtained in I

$$\langle l|qp+pq|l\rangle = 2\hbar(\gamma/\omega)(l+\tfrac{1}{2}), \qquad \langle l|K|l\rangle = \hbar\omega(l+\tfrac{1}{2}),$$

follow from equations (3.16) on making the replacements $\Omega\to\omega$, $\gamma\tan\nu t\to\gamma$.

4. Solution of the Heisenberg equations

Using the Hamiltonian given by equations (3.3) and (3.5)

$$H_0 = \tfrac{1}{2}p_0^2/M_0 + \tfrac{1}{2}M_0\omega_0^2q_0^2 - \tfrac{1}{2}\nu\tan\nu t(q_0p_0+p_0q_0), \tag{4.1}$$

we find the quantum or classical equations of motion

$$\mathrm{d}q_0/\mathrm{d}t = p_0/M_0 - \nu q_0\tan\nu t, \qquad \mathrm{d}p_0/\mathrm{d}t = \nu p_0\tan\nu t - M_0\omega_0^2q_0, \tag{4.2a}$$

or

$$\ddot{q}_0 + \Omega^2 q_0 = 0, \qquad \ddot{p}_0 + (\Omega^2 - 2\nu^2\sec^2\nu t)p_0 = 0. \tag{4.2b}$$

Equations (4.2a) are the same as in I with $\gamma\to\nu\tan\nu t$, and Q, P replaced by q_0, p_0. This identification is lost in equations (4.2b). On reverting to the original coordinate and momentum by the transformation (3.4) and using the variable mass, the solutions of equations (4.2) are

$$q(t) = q(0)\sec\nu t\cos\Omega t + p(0)\cos\nu t\sin\Omega t/[\Omega M(t)], \tag{4.3a}$$

$$p(t) = p(0)\cos\nu t[(\nu/\Omega)\tan\nu t\sin\Omega t + \cos\Omega t] + q(0)\Omega M(t)\sec\nu t[(\nu/\Omega)\tan\nu t\cos\Omega t - \sin\Omega t]. \tag{4.3b}$$

The result (4.3a) is especially simple, being the constant mass equation with $M_0\to M(t)$, as we recognise when we write it in the form

$$q_0(t) = q_0(0)\cos\Omega t + (p_0(0)/\Omega M_0)\sin\Omega t, \tag{4.3c}$$

which is actually simpler than the corresponding result (4.3a) in I. This is because

$\gamma_0(0)=0$. Not surprisingly, each of the expressions (4.3*a*, *b*) becomes infinite whenever $M=0$.

From equations (4.3*a*, *b*), or directly from the Heisenberg equations, we find the following expressions for $V(t)$ and $T(t)$:

$$V(t)=(\omega_0/\Omega)^2 T(0)\sin^2\Omega t+V(0)\cos^2\Omega t+\tfrac{1}{2}(\omega_0^2/\Omega)[q(0),p(0)]_+\sin\Omega t\cos\Omega t, \tag{4.4a}$$

$$\begin{aligned}T(t)=(\Omega/\omega_0)^2 V(0)&[(\nu/\Omega)\tan\nu t\cos\Omega t-\sin\Omega t]^2\\ &+T(0)[(\nu/\Omega)\tan\nu t\sin\Omega t+\cos\Omega t]^2\\ &+\tfrac{1}{2}\Omega[q(0),p(0)]_+[(\nu/\Omega)\tan\nu t\cos\Omega t-\sin\Omega t]\\ &\times[(\nu/\Omega)\tan\nu t\sin\Omega t+\sin\Omega t].\end{aligned} \tag{4.4b}$$

Adding these, we obtain

$$\begin{aligned}T+V=(\Omega/\omega_0)^2 V(0)&[1-(\nu/\Omega)\tan\nu t\sin 2\Omega t+(\nu/\Omega)^2(\tan^2\nu t-1)\cos^2\Omega t]\\ &+T(0)[1+(\nu/\Omega)\tan\nu t\sin 2\Omega t+(\nu/\Omega)^2(\tan^2\nu t-1)\sin^2\Omega t]\\ &+\tfrac{1}{2}\Omega[q(0),p(0)]_+[(\nu/\Omega)\tan\nu t\cos 2\Omega t\\ &+\tfrac{1}{2}(\nu/\Omega)^2(\tan^2\nu t-1)\sin 2\Omega t].\end{aligned} \tag{4.5}$$

We may easily check that the expectation values (3.16*a*, *b*) follow from (4.4*a*, *b*). We remarked in § 3 that the expectation values (3.16) could be obtained from the corresponding expressions in I by the replacement $\omega\to\Omega$, $\gamma\to\nu\tan\nu t$. This does *not* apply for the expressions (4.4*a*, *b*).

5. The energy operator

A classical analogue of the system described by equations (2.1) and (3.1) consists of a particle of mass $M(t)$ attracted to $q=0$ by a force $M(t)\omega_0^2 q$. The mass given by equation (2.1) may be supposed periodically to condense from and return to a dust cloud at rest. The equation of motion is

$$\ddot{q}-2\nu\tan\nu t\dot{q}+\omega_0^2 q=0, \tag{5.1}$$

or, with $q_0=q\cos\nu t$ as in equation (3.4),

$$\ddot{q}_0+(\omega_0^2+\nu^2)q_0=0. \tag{5.2}$$

This agrees with the Heisenberg equation given in (4.2).

Since the force field is that for the constant mass harmonic oscillator multiplied by the factor $\cos^2\nu t$, we may conclude that the energy of the classical or quantum system is also multiplied by this factor, i.e.

$$E=\cos^2\nu t(T+V). \tag{5.3}$$

Alternatively, we may arrive at this definition of the energy operator by considering the classical energy of a cavity field, as in I. Equation (5.3) is a natural extension of the definition of the energy in the case of a decaying oscillator (Hasse 1975, Tartaglia 1977). We see from equations (3.16*a*, *b*) and (5.3) that the expected value of the energy is always finite; thus

$$\langle l|E|l\rangle = \hbar\Omega[\cos^2 \nu t - \tfrac{1}{2}(\nu/\Omega)^2 \cos 2\nu t](l+\tfrac{1}{2}). \tag{5.4}$$

Maximum values occur at times when $M = M_0$,

$$\langle E\rangle_{\max} = \hbar\Omega[1-\tfrac{1}{2}(\nu/\Omega)^2](l+\tfrac{1}{2}), \tag{5.5a}$$

and minima occur whenever $M = 0$,

$$\langle E\rangle_{\min} = \tfrac{1}{2}\hbar\Omega(\nu/\Omega)^2(l+\tfrac{1}{2}). \tag{5.5b}$$

An interesting relation follows from equation (3.16*b*) as $\omega_0 \to 0$, when we obtain for a free pulsating particle

$$\mathrm{d}\langle T\rangle_0/\mathrm{d}t = 2\nu \tan \nu t \langle T\rangle_0, \tag{5.6}$$

which integrates immediately to give

$$\langle T(t)\rangle_0 = \langle T(0)\rangle_0 \sec^2 \nu t. \tag{5.7}$$

Equation (5.6) can be compared with Hasse (1975, equation (2.9)) with $\gamma \to \nu \tan \nu t$. Our result (5.7) coupled with equation (5.3) ensures conservation of energy for the free pulsating particle.

6. Dirac operators

The dimensionless coordinate and momentum

$$Q_0 = [M_0\omega_0/\hbar]^{1/2} q_0 = [M(t)\omega_0/\hbar]^{1/2} q, \tag{6.1a}$$

$$P_0 = [M_0\omega_0\hbar]^{-1/2} p_0 = [M(t)\omega_0\hbar]^{-1/2} p, \tag{6.1b}$$

give rise to Dirac operators in the usual manner:

$$A_0 = 2^{-1/2}(Q_0 + \mathrm{i}P_0), \qquad A_0^\dagger = 2^{-1/2}(Q_0 - \mathrm{i}P_0). \tag{6.2}$$

Equation (3.3) may be written

$$H_0 = \tfrac{1}{2}\hbar\omega_0(Q_0^2 + P_0^2) - \tfrac{1}{2}\hbar\nu \tan \nu t(Q_0P_0 + P_0Q_0) \tag{6.3a}$$

$$= \hbar\omega_0(A_0^\dagger A_0 + \tfrac{1}{2}) - \tfrac{1}{2}\mathrm{i}\hbar\nu \tan \nu t(A_0^{\dagger 2} - A_0^2). \tag{6.3b}$$

Let us introduce an explicitly time-dependent operator

$$A_1(t) = (2M_0\Omega\hbar)^{-1/2}[M_0(\Omega - \mathrm{i}\nu \tan \nu t)q_0 + \mathrm{i}p_0] \tag{6.4a}$$

or, in terms of the original physical coordinate and momentum and the actual mass,

$$A_1(t) = [2M(t)\Omega\hbar]^{-1/2}[M(t)(\Omega - \mathrm{i}\nu \tan \nu t)q + \mathrm{i}p]. \tag{6.4b}$$

Obviously $A_1(t)$ and its adjoint satisfy the canonical relation

$$[A_1(t), A_1^\dagger(t)] = 1, \tag{6.5}$$

and from the Heisenberg equations (4.3) it easily follows that

$$A_1(t) = A_1(0)\exp(-\mathrm{i}\Omega t), \qquad A_1^\dagger(t) = A_1^\dagger(0)\exp(\mathrm{i}\Omega t). \tag{6.6}$$

H_0 may be expressed in terms of A_1, $A_1^\dagger$ in the form

$$H_0 = \hbar\Omega(A_1^\dagger A_1 + \tfrac{1}{2}) - \tfrac{1}{4}\hbar(\gamma^2/\Omega)\sec^2 \nu t(A_1 + A_1^\dagger)^2 \tag{6.7}$$

or, on introducing a new dimensionless coordinate Q_1 and momentum P_1 according to

$$Q_1 = 2^{-1/2}(A_1 + A_1^\dagger), \qquad \mathrm{i}P_1 = 2^{-1/2}(A_1 - A_1^\dagger), \tag{6.8}$$

we may write

$$H_0 = \tfrac{1}{2}\hbar\Omega(Q_1^2 + P_1^2) - \tfrac{1}{2}\hbar(\nu^2/\Omega)\sec^2 \nu t Q_1^2. \tag{6.9}$$

We note that H_0 is *not* the canonical Hamiltonian in the variables Q_1 and P_1. This will be found in § 7.

7. The diagonalising transformation

The canonical transformation $(A_0, A_0^\dagger) \rightarrow (A_1, A_1^\dagger)$ described by equations (6.1), (6.2) and (6.4) is equivalent to $(Q_0, P_0) \rightarrow (Q_1, P_1)$, where

$$\begin{pmatrix} Q_1 \\ P_1 \end{pmatrix} = (\Omega\omega_0)^{-1/2}\begin{pmatrix} \Omega & 0 \\ -\nu \tan \nu t & \omega_0 \end{pmatrix}\begin{pmatrix} Q_0 \\ P_0 \end{pmatrix}. \tag{7.1a}$$

On referring back to the physical coordinate q and momentum p, the transformation becomes

$$\begin{pmatrix} Q_1 \\ P_1 \end{pmatrix} = (\hbar\Omega)^{-1/2}\begin{pmatrix} \Omega M^{1/2}(t) & 0 \\ -\nu \tan \nu t M^{1/2}(t) & M^{-1/2}(t) \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}. \tag{7.1b}$$

In the notation of Goldstein (1980), the generating function is

$$F_2(Q_0, P_1, t) = \tfrac{1}{2}(\nu/\omega_0)\tan \nu t Q_0^2 + \tfrac{1}{2}(\Omega/\omega_0)^{1/2}(Q_0 P_1 + P_1 Q_0), \tag{7.2}$$

and the new canonical Hamiltonian is

$$H_1(Q_1, P_1, t) = H_0(Q_1, P_1, t) + \hbar\partial F_2/\partial t, \tag{7.3}$$

i.e. we must add the term $\hbar\partial F_2/\partial t$ to the expression given in equation (6.9). Thus

$$H_1 = \tfrac{1}{2}\hbar\Omega(Q_1^2 + P_1^2) - \tfrac{1}{2}(\nu^2/\Omega)\sec^2 \nu t Q_1^2 + \tfrac{1}{2}\hbar(\nu^2/\omega_0)\sec^2 \nu t Q_0^2. \tag{7.4}$$

When we recall that by equation (7.1a) $Q_1 = (\Omega/\omega_0)^{1/2}Q_0$, we see that the last two terms in equation (7.4) cancel and we are left with the remarkably simple result

$$\begin{aligned} H_1 &= \tfrac{1}{2}\hbar\Omega(Q_1^2 + P_1^2), \qquad [Q_1, P_1] = \mathrm{i}, \\ H_1 &= \hbar\Omega(A_1^\dagger A_1 + \tfrac{1}{2}), \qquad [A_1, A_1^\dagger] = 1. \end{aligned} \tag{7.5}$$

The original problem (3.1) is thus solved by applying the transformation (7.1b) to the solution of the time-free system (7.5).

8. The Green function

Since the Heisenberg equation (4.3c) is exactly the same as for constant mass with $\omega_0 \rightarrow \Omega$, the Green function in q_0 is closely related to the usual constant mass expression;

thus with normalisation over $(-\infty, \infty)$ in q_0

$$G(q_0, q_0', t) = \left(\frac{M_0\Omega}{2\pi\hbar|\sin\Omega t|}\right)^{1/2} \exp\left(\frac{iM_0\Omega}{2\hbar\sin\Omega t}[(q_0^2+q_0'^2)\cos\Omega t + (\nu/\Omega)\tan\nu t\sin\Omega t\, q_0'^2 - 2q_0q_0']\right). \tag{8.1}$$

An extra term has appeared in the exponent owing to the asymmetry of equations (4.3a,b). We note that in equation (8.4) in I the coefficient of Q_0^2 in the exponent should be $\cos\omega t + (\gamma/\omega)\sin\omega t$ (cf Landowitz et al 1979), also we note that in I the Green function $G(Q, Q_0, t)$, like the wavefunction $\psi_l(Q)$ was normalised in Q rather than the physical coordinate q.

Changing to the physical coordinate q by the transformation (3.2) and using a normalisation over $(-\infty, \infty)$ in q, we may rewrite equation (8.1) in the form

$$G(q, q', t) = \left(\frac{M(t)\Omega}{2\pi\hbar|\sin\Omega t|}\right)^{1/2} \exp\left(\frac{iM(t)\Omega}{2\hbar\sin\Omega t}[(q^2+q'^2\sec^2\nu t)\cos\Omega t + (\nu/\Omega)\tan\nu t\sec^2\nu t\sin\Omega t\, q'^2 - 2qq'\sec\nu t]\right) \tag{8.2}$$

where $q' = q(0)$.

9. Quasi-coherent states

The well known harmonic oscillator theory holds in the representation (7.5). For instance, number states $|l\rangle$ exist which give rise to the pseudostationary states $\psi_l(q, t) = \langle q|l\rangle$ considered in § 3. Again, coherent states may be constructed which satisfy

$$A_1(t)|\alpha t\rangle = \alpha(t)|\alpha t\rangle, \tag{9.1a}$$

where by equation (6.6)

$$\alpha(t) = \alpha(0)\exp(-i\Omega t). \tag{9.1b}$$

Equations (4.3) and (9.1) give the expectation values and uncertainties

$$\langle\alpha t|q(t)|\alpha t\rangle = [\hbar/(2M_0\Omega)]^{1/2}[\alpha(t)+\alpha^*(t)]\sec\nu t, \tag{9.2a}$$

$$\langle\alpha t|p(t)|\alpha t\rangle = (\hbar M_0\Omega/2)^{1/2}\{(\nu/\Omega)[\alpha(t)+\alpha^*(t)]\sin\nu t - i[\alpha(t)-\alpha^*(t)]\cos\nu t\}, \tag{9.2b}$$

$$\Delta q(t) = [\hbar/(2M_0\Omega)]^{1/2}\sec\nu t, \tag{9.2c}$$

$$\Delta p(t) = (\hbar M_0\Omega/2)^{1/2}[(\nu/\Omega)^2\sin^2\nu t + \cos^2\nu t]^{1/2}. \tag{9.2d}$$

We note that the uncertainty in q is infinite whenever $M = 0$, but the uncertainty in p is always finite. The product of uncertainties is

$$\Delta q\Delta p = \tfrac{1}{2}\hbar[1+(\nu/\Omega)^2\tan^2\nu t]^{1/2}. \tag{9.3}$$

Coherence is lost to some extent whenever $M \neq M_0$, but as it is periodically regained we may refer to the solutions of equation (9.1) as quasi-coherent states.

In the Schrödinger representation the quasi-coherent state equation (9.1) becomes

$$\left(\frac{d}{dq_0} + \frac{M_0}{\hbar}(\Omega - i\nu\tan\nu t)q_0\right)\psi_\alpha(q_0, t) = \alpha\left(\frac{2\Omega M_0}{\hbar}\right)^{1/2}\psi_\alpha(q_0, t). \tag{9.4}$$

In terms of $q(t)$ the normalised solution is

$$\psi_\alpha(q, t) = N_\alpha(t) \exp\{\alpha[2M(t)\Omega/\hbar]^{1/2} q - M(t)/(2\hbar)(Q - \mathrm{i}\nu \tan \nu t) q^2\}. \tag{9.5a}$$

where

$$N_\alpha(t) = [\Omega M(t)/(\pi\hbar)]^{1/4} \exp\{-\tfrac{1}{4}[\alpha(t) + \alpha^*(t)]^2\}. \tag{9.5b}$$

Alternatively, we may write (cf Louisell 1973, p 109)

$$\psi_\alpha(q, t) = [M(t)\Omega/(\pi\hbar)]^{1/4} \times \exp\left(-\frac{M(t)\Omega}{2\hbar}(q - \langle\alpha|q|\alpha\rangle)^2 + \frac{\mathrm{i}q}{\hbar}[\langle\alpha|p|\alpha\rangle - M(t)\nu \tan \nu t(\langle\alpha|q|\alpha\rangle - \tfrac{1}{2}q)]\right), \tag{9.6}$$

where $\langle\alpha|q|\alpha\rangle$ and $\langle\alpha|p|\alpha\rangle$ are given by equations (9.2a, b). Thus the coordinate distribution has the usual form

$$|\psi_\alpha(q, t)|^2 = [2\pi(\Delta q)^2]^{-1/2} \exp[(q - \langle\alpha|q|\alpha\rangle)^2/2(\Delta q)^2], \tag{9.7}$$

i.e. a Gaussian of half-width Δq, centred at $q = \langle\alpha|q|\alpha\rangle$, but here Δq and $\langle\alpha|q|\alpha\rangle$ vary with the time according to equations (9.2a, c). The connection with the pseudostationary states (3.8) is, as discussed by Dodonov and Man'ko (1979),

$$\psi_\alpha(q, t) = \exp(-\tfrac{1}{2}|\alpha|^2) \sum_{l=0}^{\infty} \alpha^l (l!)^{-1/2} \psi_l(q, t), \tag{9.8}$$

where

$$\psi_l(q, t) = (2^l l!)^{-1/2} H_l([M(t)\Omega/\hbar]^{1/2} q) \psi_0(q, t). \tag{9.9}$$

10. The connection between the Schrödinger q and Q_1 representations

We recall that in § 3 we defined $q_0 = q \cos \nu t$ and in § 6 $Q_0 = (M_0\omega_0/\hbar)^{1/2} q_0$. The solutions $\langle q_0|l\rangle$ of the Schrödinger equation (3.6) are trivially transformed to the solutions of the equation

$$\tfrac{1}{2}[(\omega_0/\Omega)(Q_0^2 - \partial^2/\partial Q_0^2) + \mathrm{i}(\nu/\Omega) \tan \nu t(2Q_0\partial/\partial Q_0 + 1)]\langle Q_0|l\rangle = (l + \tfrac{1}{2})\langle Q_0|l\rangle. \tag{10.1}$$

In § 7 we defined $Q_1 = (\Omega/\omega_0)^{1/2} Q_0$ and at first sight it might appear that the connection between the wavefunctions $\langle Q_1|l\rangle$ and $\langle Q_0|l\rangle$ is also trivial. However, this is not the case since $\langle Q_1|l\rangle$ satisfies the very different Schrödinger equation associated with equation (7.5):

$$\tfrac{1}{2}(Q_1^2 - \partial^2/\partial Q_1^2)\langle Q_1|l\rangle = (l + \tfrac{1}{2})\langle Q_1|l\rangle. \tag{10.2}$$

To change from (10.1) to (10.2) we have to follow through the transformation theory of § 7. From equation (7.1a)

$$-\mathrm{i}\partial/\partial Q_1 = -\nu \tan \nu t(\Omega\omega_0)^{-1/2} Q_0 + (\omega_0/\Omega)^{1/2}(-\mathrm{i}\partial/\partial Q_0). \tag{10.3}$$

Using equations (7.3) and (10.3), we may change the Hamiltonian

$$H_1 = \tfrac{1}{2}\hbar\Omega(Q_1^2 - \partial^2/\partial Q_1^2) \tag{10.4}$$

of equation (10.2) into the form

$$H_0 = \tfrac{1}{2}\hbar\omega_0(Q_0^2 - \partial^2/\partial Q_0^2) + i\hbar\nu \tan \nu t(Q_0\partial/\partial Q_0 + \tfrac{1}{2}) \tag{10.5}$$

needed for equation (10.1). If future solvable time-dependent problems are forthcoming, we could perhaps avoid solving complicated Schrödinger equations by transforming the constant mass harmonic oscillator wavefunctions $\langle Q_1|l\rangle$ directly using

$$\langle Q_0|l\rangle = \int_{-\infty}^{\infty} dQ_1 \langle Q_0|Q_1\rangle\langle Q_1|l\rangle. \tag{10.6}$$

However, it is not at all easy to see how to calculate the kernel functions $\langle Q_0|Q_1\rangle$ directly from canonical transformation equations of the type (7.1).

11. Conclusion

The most significant effect of pulsating mass in a harmonic oscillator is to change the natural frequency ω_0 to an effective frequency Ω given by equation (1.3). This effective frequency follows from the solution in I by the replacement $\gamma \to i\Omega$, causing $\omega \to \Omega$, ω being given by equation (1.2). For the wavefunction and expectation values given in § 3 the replacement $\gamma \to \nu \tan \nu t$ is necessary in addition to $\omega \to \Omega$. However, we see that any such algebraic replacement is inadequate to describe the underlying dynamics as exhibited in the Heisenberg equations and their solutions in § 4. On the other hand, the Green function originating in a delta function distribution at $t = 0$ is the constant mass result with an extra oscillatory exponential factor. Also we have shown the existence of quasi-coherent states which periodically return to minimum uncertainty wavepacket solutions.

The comparisons which we are able to make with the work of Dodonov and Man'ko (1979) are pleasing. These authors employed a mass of the type $M_0 \exp[2\Gamma(t)]$ and our work can be regarded as an analytic continuation of the form $\Gamma(t) \to i\nu t$, but by using real mass we have avoided any difficulties that might arise from a non-Hermitian Hamiltonian.

We have put forward in equation (5.3) a definition of the energy operator which is justified by the consideration of a classical analogue. This ensures conservation of the energy of a free pulsating mass. In §§ 3, 4 we have given separate results for T and V (essentially the kinetic and potential energies) because we think it is interesting to see how V depends on ν only through Ω, whereas T takes up a stronger dependence on ν.

We remarked in § 3 that there is no obvious extension of our solution for another power of $\cos \nu t$ in the mass law (2.1). We should prefer a gentler variation of mass with time that avoids the periodic $M \to 0$. Whether it is possible to reduce further problems to a time-free representation of the form (7.5) is an open question. If it is possible, then our remarks below on possible applications became even more cogent. Perhaps we cannot expect the simplicity of the present solution to be repeatable. Further progress may have to depend on approximate and numerical methods (Colegrave and Abdalla 1982).

This second solvable case of a harmonic oscillator with mass varying according to equation (2.1) is certainly no less important than the first case of exponentially varying mass reported in I. We hope that the term pulsating mass may stimulate some possible applications. Our particular interest is in laser-driven or laser-producing cavities (Colegrave and Abdalla 1981a), and we feel that the present solution and possible future extensions could be of paramount importance in quantum optics and perhaps in other branches of physics including solid state physics and quantum field theory. The Heisenberg solution in § 4 is formally the same as the classical solution, and as such it could have applications in the classical regime.

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